Noise and synchronization in chaotic systems

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(Received 9 March 1995; revised manuscript received 3 November 1995)

We investigate the behavior of several noisy nonlinear dynamical models in order to find out whether the presence of a common noise term may synchronize identical chaotic systems as recently supposed [S. Fahy and D. R. Hamann, Phys. Rev. Lett. **69**, 761 (1993); A. Maritan and J. R. Banavar, *ibid.* **72**, 1451 (1994)]. The results of the present study show that noise can speed up orbit convergence in a restricted context, but in general cannot drive, by itself, a transition from chaotic to nonchaotic behavior. [S1063-651X(96)01406-7]

PACS number(s): 05.45.+b

Nonlinear dynamical systems in a chaotic regime are characterized by sensitive dependence on the initial conditions [1]. Since the deterministic evolution of such systems can display an apparently random character, *a fortiori* one expects that in the presence of truly random forces the behavior of a chaotic system becomes even more ''random.'' Counter to this intuitive expectation, some recent results [2,3] concerning Brownian-type motions in a confining potential as well as some properly called chaotic systems have been interpreted as evidence that noise may drive a transition of nonlinear dynamical systems from a chaotic to nonchaotic behavior, in the sense that the final trajectory is completely independent of the initial conditions. In order to assess the validity of this conclusion, we investigate in this paper the behavior of several types of noisy nonlinear systems. The results hereby obtained show no evidence that noise can induce *by itself* synchronization of chaotic systems. On the contrary, they make clear that the explanation of the behavior observed in the literature [2,3] is to be found, as detailed in the following, in other causes such as dissipation or biases affecting noise. We observed, nevertheless, an important effect of noise (though less dramatic than synchronization of the trajectories): orbit convergence for particles subjected to dissipation and to a nonlinear confining force is speeded up by the presence of noise, provided that its amplitude is small enough with respect to dissipation.

Let us first consider a particle moving in a potential $U(x)$, subject to a dissipation proportional to its velocity and to a noise term

$$\ddot{x}=F(x)-\alpha\dot{x}+\Gamma\,\eta_{sym}, \tag{1}$$

where $F(x)=-dU(x)/dx$, α is the dissipation coefficient, Γ is the noise amplitude, and η_{sym} is a random number chosen uniformly out of the interval $[-1,1]$. We note that the noise term is a symmetrical one, in the sense that its mean value is zero. We exploited a numerical solution of Eq. (1), using a fourth-order Runge-Kutta scheme in which noise is added at each interpolation point after scaling its amplitude with the step size of the integration. Calculations were carried out in double precision [4] with a time step of 0.001 (test runs were also performed with a time step of 0.0001). Upon choosing $U(x)=x^4$ (the one-dimensional Duffing potential), we investigated orbit convergence by calculating the mean square distance σ, in the $(x,\dot{x})$ phase space, between pairs of particles obeying Eq. (1) (with the *same* noise), averaged over several (100) randomly chosen initial conditions.

Figure 1 shows σ as a function of the iteration number for $\alpha=0.1$ and for different values of Γ. When $\Gamma=0$, the origin $(x=0,\ \dot{x}=0)$ is a point attractor for the system considered; the asymptotic convergence rate is not very high: hence the slow decrease of σ is hardly appreciable in the figure. As soon as noise is switched on, this attractor loses its stability and the orbits coalesce for long times into a single random orbit independent of the initial conditions. We note that the rate of convergence is considerably higher than that of the nondriven damped system. For example, at a noise amplitude $\Gamma=0.2$, after 1000 time steps, σ is about six orders of magnitude smaller than when noise is absent. As Γ increases, the convergence rate initially increases (up to $\Gamma\sim1$), but then orbit convergence slows down until, for Γ greater than a threshold value Γ_c, the rate of convergence of the noisy system becomes smaller that that of the nondriven oscillator (as shown in Fig. 1, for $\alpha=0.1$ $20<\Gamma_c<30$). This inversion monitors the transition to chaotic behavior: in fact as soon as Γ increases above Γ_c, synchronization is no longer observed (σ does not go to zero) and the final orbit depends randomly on the initial conditions. Upon calculating Γ_c for different values of α, we were able to locate approximately in the (α,Γ) plane the regions corresponding to chaotic and to non-

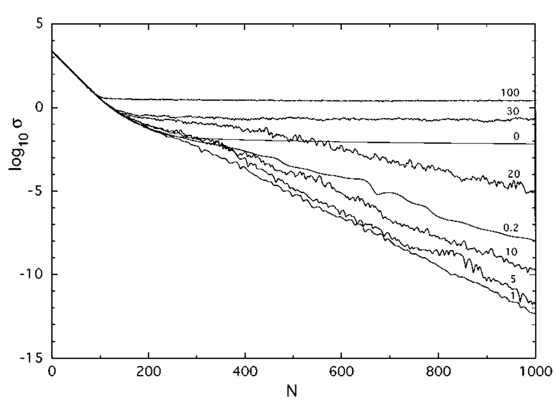

FIG. 1. Mean square distance versus number of iterations (each unit represents 1000 iterations) between pairs of particles obeying Eq. (1) for $\alpha=0.1$. Each curve is labeled with the corresponding value of Γ. Results were averaged over 100 randomly chosen initial conditions.

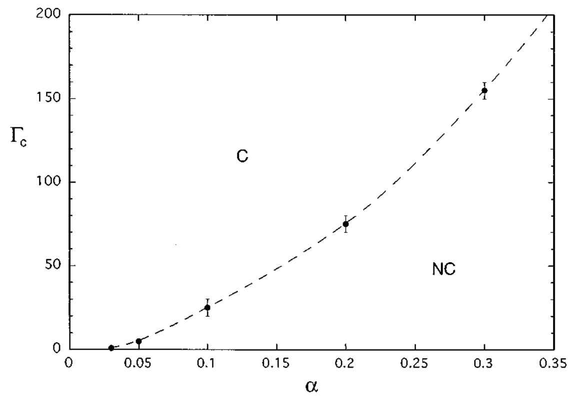

FIG. 2. Plot of Γ_c (defined in the text) versus α. The error bar denotes the step used to increment Γ when localizing Γ_c. At $\alpha = 0.03$ and $\alpha = 0.05$ this step is smaller than the dot dimension. The labels NC and C indicate regions of nonchaotic and chaotic behavior, respectively.

chaotic behavior, respectively (see Fig. 2). Though, as already remarked, the presence of noise in the nonchaotic region speeds up orbit convergence with respect to the nonnoisy case (the $\Gamma = 0$ axis), it appears altogether evident that the key role in driving the transition from chaotic to nonchaotic behavior is played by dissipation rather than by noise. In fact any path from the chaotic to the nonchaotic region implies a decrease of the noise amplitude Γ or, alternatively, an increase of Γ, provided that this effect is more than compensated by a simultaneous increase of the dissipation coefficient. Behavior qualitatively similar to that reported above has been observed for a variety of confining one-dimensional potentials [5].

Let us now reconsider, in the light of the present findings, the results obtained by Fahy and Hamann [2]. These authors, using a molecular dynamics approach, showed that particles in a fixed external potential, driven by an identical sequence of random forces (particles are stopped and restarted with random velocities at regular intervals of length τ), follow random trajectories which become identical for long times if τ is small enough. In Ref. [2] this transition from chaotic to nonchaotic behavior was attributed to the enhancement of the random perturbation of the system resulting from a decrease of τ. However, such an interpretation may be too simplistic. In fact, the action of stopping and restarting the particles can be thought of as an infinite damping force turned on for an infinitesimal time at regular intervals, followed by a random driving force with the appropriate impulse [6]. Reducing τ has then the double effect of increasing noise and dissipation. It thus follows that the observed synchronization cannot be attributed solely to noise. On the other hand, since at variance with our model system it is impossible to disentangle the two effects, one cannot study the roles played by noise and dissipation, respectively.

The system studied in Ref. [2] was examined also by Maritan and Banavar [3], who noted that, in the limit of small τ, it can be understood in terms of the Langevin dynamics [7]. They showed that particles moving in a fixed external potential and following a Langevin equation of the form $\dot{\vec{x}} = -\vec{\nabla}_x U(\vec{x}) + \vec{\eta}(t)$, with the same noise, collapse into the same trajectory at long times. This result was interpreted by Maritan and Banavar [3] as evidence of the supposed synchronizing effect of noise. These authors, however, did not thoroughly consider the implications of the above argument as far as the causes of the predicted synchronization are concerned. In fact, it is to be observed that (a) the Langevin equation, assuming that velocity is proportional to the force, is justified only for ''overdamped'' motions [7]; (b) synchronization of the trajectories is predicted *independently* of the noise amplitude. Thus, on the basis of the argument presented by Maritan and Banavar [3], one can simply conclude that when dissipation is dominant with respect to all the other forces acting on the system, synchronization is expected for any noise amplitude, even for an arbitrarily small one. This does not justify the assumption that the predicted transition is driven by the presence of noise. On the contrary, these considerations suggest, in agreement with our results, that the causes of the addressed transition from chaotic to nonchaotic behavior should be found in the dominant role of dissipation.

Let us now turn our attention to the effect of noise on the long-term behavior of properly called chaotic systems. Maritan and Banavar [3] investigated numerically two such systems (i.e., the Lorenz system and the one-dimensional logistic map), drawing the conclusion that noise may in general induce synchronization of chaotic systems. Here we will reexamine both cases in detail. We first consider the Lorenz system [1]

$$dx/dt = Px - Py\,(\mathrm{a}), \quad dy/dt = -xz + Rx - y\,(\mathrm{b}),$$
$$dz/dt = xy - Bz\,(\mathrm{c}), \tag{2}$$

with $P = 10$, $R = 28$, and $B = 8/3$. In Ref. [3] it was shown that the addition of a noise term to any of the equations of the Lorenz system causes orbit synchronization when the noise level surpasses a threshold, but only if the noise is asymmetric, i.e., of the type $\Gamma\eta_{asym}$, where Γ is the amplitude and η_{asym} is a random number chosen uniformly out of the interval [0,1]. No synchronization was observed for a symmetric noise term. Such different behavior, pointed out, but not explained, by Maritan and Banavar [3], can illuminate the origin of the observed phenomenon. In order to find out why asymmetric and symmetric noises have such different effects, we observe that choosing a number randomly from the interval [0,1] is equivalent to choosing it from the interval [−1,1], then adding 1, and dividing the result by 2. In a statistical sense we can write

$$\eta_{asym} \equiv (\eta_{sym} + 1)/2. \tag{3}$$

Thus an asymmetric noise of the type specified above is nothing else than a symmetric noise superimposed upon a constant background level. In Fig. 3 we show how the mean square distance σ between pairs of trajectories depends on time for the Lorenz system modified by adding to Eq. (2b) the term $\Gamma\eta_{asym}$, $(\Gamma/2)\eta_{sym}$, or $\Gamma/2$, respectively [8]. We chose a value of Γ great enough to attain synchronization with an asymmetric noise term. After a short initial transient, σ rapidly decreases when we add either $\Gamma\eta_{asym}$ or $\Gamma/2$: independently of the initial conditions, trajectories collapse into the same final orbit (which is random only in the first case). It is important to stress that the slope of the curve is

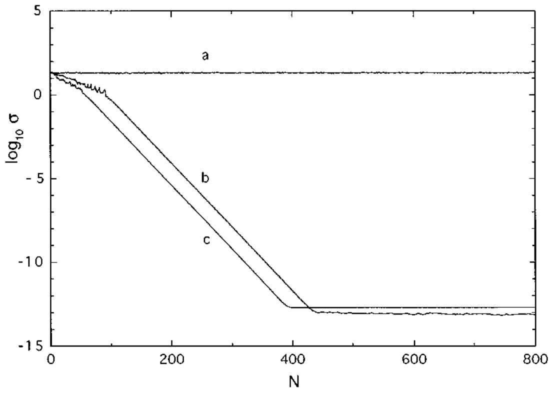

FIG. 3. Mean square distance versus number of iterations (each unit represents 1000 iterations) for the Lorenz system modified by adding to Eq. (2b), respectively, $(\Gamma/2)\eta_{sym}$ (curve a), $\Gamma\eta_{asym}$ (curve b), and $\Gamma/2$ (curve c). Results were averaged over 100 randomly chosen initial conditions. $\Gamma=20$.

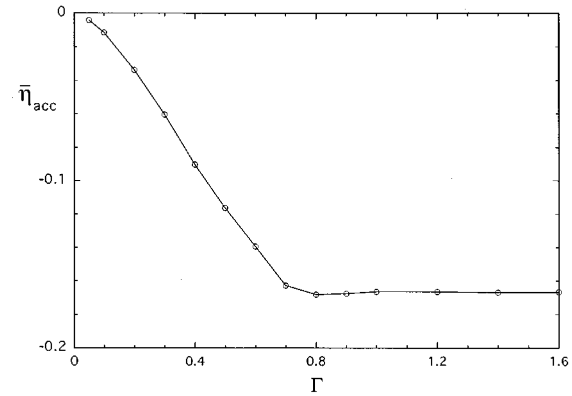

FIG. 4. $\overline{\eta_{acc}}$ versus Γ for the logistic map, with η chosen out of the interval $[-\Gamma,\Gamma]$. Results were averaged over 100 runs of 10^6 iterations starting from randomly chosen initial conditions. The line is obtained by connecting points through straight line segments.

the same in the two cases, which makes clear that the synchronizing effect of the asymmetrical noise is determined not by the random portion of the added term, which *by itself induces no synchronization* (see curve a), but by its constant background level, i.e., by the noise mean value. In order to confirm this interpretation, we studied the behavior of the Lorenz system modified by adding to Eq. (2b) the term $\Gamma\eta_{asym}$ or $\Gamma/2$, as a function of the amplitude Γ. We found that synchronization is observed when Γ is greater than a threshold value which is approximately the same in the two cases. Results qualitatively similar to those reported above have been obtained for the Duffing oscillator [5].

We consider now the noisy logistic map investigated in Ref. [3]:

$$x_{n+1}=4x_n(1-x_n)+\eta_n, \tag{4}$$

where $0<x_n<1$ and η_n is a random number chosen uniformly out of the interval $[-\Gamma,\Gamma]$ with the constraint $0<x_{n+1}<1$: if a given η_n violates this condition it is discarded and a new η_n is chosen. For $\Gamma>\Gamma_c$ ($\Gamma_c\simeq0.5$), trajectories starting from different initial conditions collapse into the same final orbit, when an identical sequence of η's is used independently of the initial condition [3]. This might appear an example of synchronization induced by a symmetrical noise. However, as already observed by Maritan and Banavar [3], the procedure adopted in order that η_n satisfies the constraint $0<x_{n+1}<1$ introduces a dependence of noise on the state of the system. It follows that the accepted η's, though belonging to a symmetrical interval, are not uniformly distributed. Indeed, as shown in Fig. 4, the average value $\overline{\eta_{acc}}$ of the accepted η's is negative for every Γ and tends to zero only for $\Gamma\to0$. It follows that, similarly to the Lorenz system, synchronization is attained by applying to the logistic map a noise with a nonzero mean value. It is to be stressed that this asymmetry, originating from the noise dependence on the dynamics of the system, stems from a deeper bias than that of a noise generated uniformly from an asymmetric interval (such as η_{asym}).

An obvious question is whether the logistic map can be synchronized by a zero-mean noise. We note that a zero-mean state-independent noise would simply bring x_n rapidly out of the basin of attraction. It is possible, however, by choosing the η's from an opportune interval, to modify $\overline{\eta_{acc}}$ so that the noise acting on the map, though state dependent, has a zero mean value. More specifically we choose the η's out of the nonsymmetrical interval $[\beta-\Gamma,\beta+\Gamma]$, where the limitation $|\beta|<\Gamma$ has to be respected since if the η's are only positive or only negative the condition $0<x_n<1$ will be rapidly violated. In Fig. 5 $\overline{\eta_{acc}}$ is shown as a function of β, for $\Gamma=0.6$. The average value of the accepted η's increases monotonically with β and vanishes around $\beta\simeq0.365$. In the same figure we show the fraction $\lambda=N_c/N$ where N is the number of different realizations (100) performed for each β, and N_c is the number of realizations for which the distance between pairs of trajectories with randomly chosen initial conditions, at the end of the run, is of the order of the numerical precision. We note that λ differs from zero inside a restricted interval centered

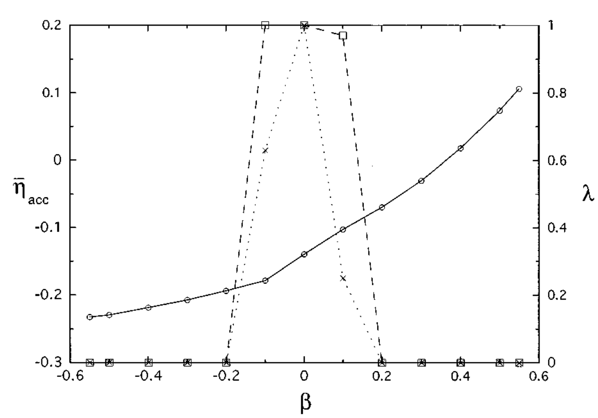

FIG. 5. $\overline{\eta_{acc}}$ (solid line) and λ (dashed and dotted lines) versus β for the logistic map, with η chosen out of the interval $[\beta-0.6,\beta+0.6]$. Results were averaged over 100 runs of 10^6 and 10^7 iterations starting from randomly chosen initial conditions. The values of $\overline{\eta_{acc}}$ are indistinguishable in the two cases, while those of λ are represented with crosses and squares, respectively. Lines are obtained by connecting points through straight line segments.

around $\beta = 0$, and its value tends to 1 in the limit of an infinite run length (as shown by the results obtained through two different sets of calculation with run lengths of 10^6 and 10^7 iterations, respectively). Outside this interval and thus also in correspondence with $\overline{\eta_{acc}} = 0$, λ is zero for both sets of calculations performed and thus no synchronization is expected to occur. The above described behavior appears to be independent of the value of Γ as confirmed by similar calculations performed for $\Gamma = 0.3$, 0.8, and 1. A more general analysis of the influence of noise on the stability properties of one-dimensional (1D) maps will be presented in the future [9]. Here, through a linear stability analysis, it is shown that the addition to a generic chaotic 1D map with a limited basin of attraction of a noise satisfying the resulting constraint does not always induce synchronization of the trajectories, and may even enhance, rather than reduce, orbital divergence.

In conclusion, the present results show no evidence that coupling nonlinear systems in a chaotic regime through identical *unbiased* noises does in general give origin to orbit synchronization. We found, however, an important effect of noise: particles subjected to dissipation and to a nonlinear confining force, when driven by an identical sequence of random forces of small amplitude with respect to dissipation, converge to a single final trajectory much faster than when noise is absent. Since noise is ubiquitous in real systems, this result would suggest that, provided that noise is not too strong, memory of the initial state is generally lost more rapidly than expected with dissipative forces only.

The author wishes to thank A. Maritan for a useful discussion. This work was supported by the Ministero dell'Università e della Ricerca Scientifica e Tecnologica and by the Consorzio Interuniversitario Nazionale per la Fisica della Materia.

[1] See, for example, H. Schuster, *Deterministic Chaos* (VCH Verlag, Weinheim, 1988).
[2] S. Fahy and D. R. Hamann, Phys. Rev. Lett. **69**, 761 (1993).
[3] A. Maritan and J. R. Banavar, Phys. Rev. Lett. **72**, 1451 (1994).
[4] On the RISC processor based computers used this corresponds to a precision of about 10^{-15}.
[5] G. Malescio (unpublished results).
[6] This consideration has appeared already in Ref. [2] but its consequences as regards the causes of the observed transition were completely neglected.
[7] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[8] Results are qualitatively independent of the equation modified.
[9] G. Malescio (unpublished).